

# Quantum uncertainty on a single observable

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Quantum mechanics predicts that measurements of incompatible observables carry an uncertainty which is independent of technical deficiencies of the measurement apparatus or incomplete knowledge of the state of the system. Nothing yet seems to prevent a single physical quantity, such as one spin component, from being measured with arbitrary precision. Here we show that an intrinsic quantum uncertainty on a single observable is ineludible in a number of physical situations. When revealed on local observables of a bipartite system, such uncertainty defines a *bona fide* and computable measure of general quantum correlations. We then demonstrate that these correlations, commonly referred to as quantum discord, constitute a resource for quantum enhanced metrology, as they can be exploited to improve the efficiency of parameter estimation with noisy probes.

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**Introduction.**— In a classical world, error bars are exclusively due to technological limitations, while quantum mechanics entails that two noncommuting observables cannot be jointly measured with arbitrary precision [1], even if one could access a flawless measurement device. The corresponding uncertainty relations have been linked to distinctive quantum features such as nonlocality, entanglement and data processing inequalities [2–4].

Remarkably, even a single quantum observable may display an intrinsic uncertainty as a result of the probabilistic character of quantum mechanics. Let us consider for instance a composite system prepared in an entangled state [5], say the Bell state  $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  of two qubits. This is an eigenstate of the global observable  $\sigma_z \otimes \sigma_z$  ( $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  are the Pauli matrices), so there is no uncertainty on the result of such a measurement. On the other hand, the measurement of *local* spin observables of the form  $\vec{a} \cdot \vec{\sigma} \otimes \mathbb{I}$  (where  $\vec{a} \neq 0$  is a real vector) is intrinsically uncertain. Indeed, the state  $|\phi^+\rangle\langle\phi^+|$ , and in general any entangled state, cannot be eigenstates of a local observable. Only uncorrelated states of the two qubits, e.g.  $|00\rangle$ , admit at least one completely ‘certain’ local observable.

Extending the argument to mixed states, one needs to filter out the uncertainty due to classical mixing, i.e., lack of knowledge of the state, in order to identify the genuinely quantum one. We say that an observable  $K$  on the state  $\rho$  is ‘quantum-certain’ when the statistical error in its measurement is solely due to classical ignorance. By adopting a meaningful quantitative definition of quantum uncertainty, as detailed later, we find that  $K$  is quantum-certain if and only if  $\rho = \rho_K$ , where  $\rho_K$  is the density matrix of the state after the measurement of  $K$ . It follows that even on unentangled states (all but a null measure set thereof [6]) no local observable can be quantum-certain. The only states left invariant by a local complete measurement are those described within classical probability theory [7], i.e., embeddings of joint probability distributions.

The quantum uncertainty on local observables is then entwined to the concept of *quantum discord* [8, 9] (see Fig. 1), which arguably represents the most general form of quantum

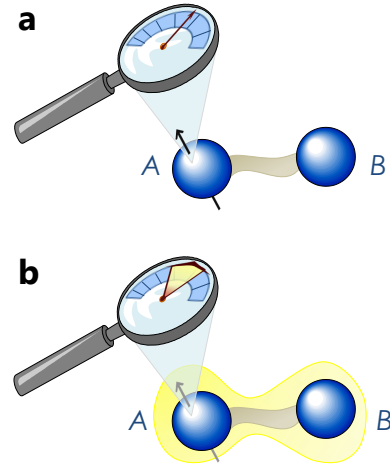


FIG. 1: (Color online) Quantum correlations trigger local quantum uncertainty. Let us consider a bipartite state  $\rho$ . An observer on subsystem A is equipped with a *quantum meter*, a measurement device whose error bar shows the quantum uncertainty only (Note: in order to access such quantity, the measurement of other observables may be required, in a procedure similar to state tomography). (a) If  $\rho$  is uncorrelated or contains only classical correlations (brown shade), i.e.  $\rho$  is of the form  $\rho = \sum_i p_i |i\rangle\langle i|_A \otimes \sigma_{iB}$  (with  $\{|i\rangle\}$  an orthonormal basis for A) [8–10], the observer can measure at least one observable on A without any intrinsic quantum uncertainty. (b) If  $\rho$  contains a nonzero amount of quantum correlations (yellow shade), i.e., entanglement for pure states [5] and quantum discord in general [10], any local measurement on A is affected by quantum uncertainty. The minimum quantum uncertainty associated to a single measurement on subsystem A quantifies the degree of quantum correlations shared in the state  $\rho$ , as perceived by the observer on A. In this work we adopt the Wigner-Yanase skew information [17] to measure the quantum uncertainty on local observables.

correlations [10], reducing to entanglement on pure states, and is currently subject to intense investigations for quantum computation and information processing [11–14]. In fact, the states  $\rho_K$  which admit a quantum-certain local observable  $K$  are the states with zero discord. In the following, a proper and

computable measure of discord is defined, interpreted and analysed within the framework of local quantum uncertainty. This allows us to show that the presence of discord in a bipartite state can tighten the Heisenberg uncertainty relation on pairs of local observables, and to establish that discord, even without entanglement, is a *resource* for noisy quantum metrology [15].

*Skew information and local quantum uncertainty.*— There are several ways to quantify the uncertainty on a measurement, and here we aim at extracting the truly quantum share. Entropic quantities or the variance, though employed extensively as indicators of uncertainty [1, 3, 4], do not fit our purpose, since they are affected by the state mixedness. It has been proposed to isolate the quantum contribution to the total statistical error of a measurement as being due to the noncommutativity between state and observable: this may be reliably quantified via the *skew information* [16, 17]

$$\mathcal{I}(\rho, K) = -\frac{1}{2} \text{Tr}[\{\rho^{\frac{1}{2}}, K\}^2], \quad (1)$$

introduced in [17] and employed for studies on uncertainty relations [16], quantum statistics and information geometry [16, 18–21]. Referring to [17] for the main properties of the skew information, we recall the most relevant ones: it is non-negative, vanishing if and only if state and observable commute, and is convex, that is, nonincreasing under classical mixing. Moreover,  $\mathcal{I}(\rho, K)$  is always smaller than the variance of  $K$ ,  $\mathcal{I}(\rho, K) \leq \text{Var}_\rho(K) \equiv \langle K^2 \rangle_\rho - \langle K \rangle_\rho^2$ , with equality reached on pure states  $\rho = |\psi\rangle\langle\psi|$ , where no classical ignorance occurs (see Fig. 2). Hence, we adopt the skew information as measure of quantum uncertainty and deliver a theoretical framework in which we convey and discuss its operational interpretation.

As a central concept in our analysis, we introduce the *local quantum uncertainty* (LQU) as the minimum skew information achievable on a single local measurement. We remark that by ‘measurement’ in this work we always refer to a complete von Neumann measurement. Let  $\rho \equiv \rho_{AB}$  be the state of a bipartite system. The LQU with respect to subsystem  $A$  is

$$\mathcal{U}_A(\rho) \equiv \min_{K^A} \mathcal{I}(\rho, K^A). \quad (2)$$

The above minimisation runs over local maximally informative observables  $K^A = K_A \otimes \mathbb{I}_B$ , where  $K_A$  is an Hermitian operator on subsystem  $A$  with nondegenerate spectrum. This restriction is necessary to exclude any epistemic ignorance on the measurement outcomes. We further note that the information gained by measuring  $K_A$  should equal that given by any classical rescaling of the form  $f(K_A)$ , with  $f$  an invertible function known to the experimenter. This defines an equivalence relation not reflected by the skew information in Eq. (2), which depends on the eigenvalues of  $K_A$  as well and not only on its eigenvectors. To overcome this, we can fix the spectrum of  $K_A$  to the simplest structure compatible with nondegeneracy, namely equispaced eigenvalues. A

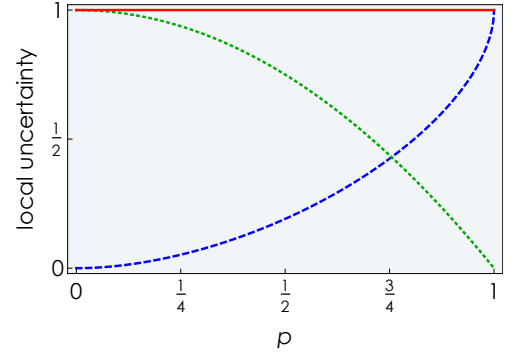


FIG. 2: (Color online) The plot shows different contributions to the error bar of spin measurements on subsystem  $A$  in a Werner state [5]  $\rho = p|\phi^+\rangle\langle\phi^+| + (1-p)\mathbb{I}/4$ ,  $p \in [0, 1]$ , of two qubits  $A$  and  $B$ . The red line is the variance  $\text{Var}_\rho(\sigma_z^A)$  of the  $\sigma_z^A$  operator, which amounts to the total statistical uncertainty. The blue dashed curve represents the local quantum uncertainty  $\mathcal{U}_A(\rho)$ , which in this case is  $\mathcal{I}(\rho, \sigma_z^A)$  (any local spin direction achieves the minimum for this class of states). The green dotted curve depicts the (normalised) linear entropy  $S_L(\rho) = \frac{4}{3}(1 - \text{Tr}[\rho^2])$  of the global state  $\rho$ , which measures its mixedness. Notice that the Werner state is separable for  $p \leq 1/3$  but it always contains discord for  $p > 0$ .

convenient choice is  $K_A \propto V_A Z_A V_A^\dagger$ , where  $V_A$  and  $Z_A$  are respectively a generic unitary and an arbitrary component of a spin- $\frac{d_A-1}{2}$  operator, with  $d_A$  the dimension of subsystem  $A$ . The so-constrained observables  $K$  describe nontrivial local Hamiltonians on the measured subsystem [22].

*Computable measure of quantum correlations.*— What characterises quantum correlations in a state is, as anticipated, the non-existence of quantum-certain local observables. In fact, we find that the quantity  $\mathcal{U}_A(\rho)$  defined in Eq. (2) is not only an indicator, but also a full fledged *measure* of bipartite quantum correlations (see Fig. 1) [23], i.e. it meets all the known *bona fide* criteria for a discord-like quantifier [10]. Specifically, in the Appendix we prove that the LQU is invariant under local unitary operations, is nonincreasing under local operations on  $B$ , vanishes if and only if  $\rho$  is a zero discord state with respect to measurements on  $A$ , and reduces to an entanglement monotone when  $\rho$  is a pure state.

We further find that quantifying discord via the LQU is very advantageous in practice, compared to all the other measures proposed in the literature (which typically involve formidably hard optimisations) [10, 24]. Indeed, the minimisation in Eq. (2) can be solved analytically so that  $\mathcal{U}_A$  admits a *computable* closed form for arbitrary states of a qubit-qudit system. Let  $\rho_{AB}$  be a state on  $\mathbb{C}^2 \otimes \mathbb{C}^d$ . The observables  $K_A$  on the qubit  $A$  take the form  $K_A = V_A \sigma_{z_A} V_A^\dagger = \vec{n} \cdot \vec{\sigma}_A$ , with  $|\vec{n}| = 1$ . Eq. (2) can thus be rewritten as a minimisation over the two angles defining  $\vec{n}$ , yielding simply

$$\mathcal{U}_A(\rho_{AB}) = 1 - \lambda_{\max}\{W_{AB}\}, \quad (3)$$

where  $\lambda_{\max}$  indicates the maximum eigenvalue, and  $W_{AB}$  is a

$3 \times 3$  symmetric matrix whose elements are

$$(W_{AB})_{ij} = \text{Tr} \left\{ \rho_{AB}^{1/2} (\sigma_{iA} \otimes \mathbb{I}_B) \rho_{AB}^{1/2} (\sigma_{jA} \otimes \mathbb{I}_B) \right\},$$

with  $i, j = x, y, z$ . It is easy to check that, for a pure state  $|\psi_{AB}\rangle\langle\psi_{AB}|$ , Eq. (3) reduces to the linear entropy of entanglement,  $\mathcal{U}_A(|\psi_{AB}\rangle\langle\psi_{AB}|) = 2(1 - \text{Tr} \rho_A^2)$ , where  $\rho_A$  is the marginal state of subsystem A. Even for general states of larger systems, the evaluation of the LQU remains a computationally tractable problem. In fact, it can be recast as a minimisation with respect to a finite number of variables spanning a compact space. This follows by noting that the unitary matrix  $V_A$  entering the definition of the observable  $K_A$  can be chosen within the special unitary group, without loss of generality. The evaluation of the LQU for Werner states of two qubits is displayed in Fig. 2. A case study of the DQC1 model of quantum computation [25] is reported in the Supplemental Material [26], showing that our measure exhibits the same scaling as the canonical entropic measure of discord [8, 11].

More generally, the adopted approach provides a clear-cut interpretation of what discord is: a local quantum feature, i.e., the minimum quantum contribution to statistical uncertainty, which mirrors a nonlocal one, i.e., correlations. Quantum uncertainty on a single local observable thus stands as a *founding principle* for general quantum correlations.

*Interplay with Heisenberg uncertainty relations.*— It is interesting to discuss the connections between the quantum uncertainty on a single observable and conventional Heisenberg uncertainty relations on pairs of incompatible observables [1]. Let us consider a bipartite state  $\rho_{AB}$  and let us pick a set of  $n$  local observables  $\{K_j^A\}$  on A, with  $j = 1, \dots, n$ . Then by definition of skew information and by using Eq. (2) we have  $\prod_j \text{Var}_{\rho_{AB}}(K_j^A) \geq \prod_j \mathcal{I}(\rho_{AB}, K_j^A) \geq [\mathcal{U}_A(\rho_{AB})]^n$ . In the specific case of a pair of local measurements,  $n = 2$ , the previous inequality can be regarded as an alternative uncertainty relation, arising from the quantum correlations in the state rather than from the noncommutativity of the chosen observables. This induces a refinement of the Heisenberg principle on pairs of local observables, which can be written in general as

$$\text{Var}_{\rho_{AB}}(K^A) \text{Var}_{\rho_{AB}}(L^A) \geq \max \left\{ \frac{1}{4} |\langle [K^A, L^A] \rangle_{\rho_{AB}}|^2, \mathcal{U}_A^2(\rho_{AB}) \right\}. \quad (4)$$

For all the bipartite states and pairs of local observables  $K^A, L^A$  such that the amount of quantum correlations on A, measured by the LQU, is strong enough, namely  $\mathcal{U}_A \geq \frac{1}{2} |\langle [K^A, L^A] \rangle|$ , then such amount (squared) yields a tighter quantitative bound on the product of the variances of the chosen observables, compared to the Heisenberg one. This is the case, for instance, for pairs of local spin measurements on two-qubit Werner states (see Fig. 2), for which the Heisenberg bound is trivially vanishing while the LQU is nonzero and increases with the purity of the state.

*Applications to noisy quantum metrology.*— We now discuss the operative role of the LQU for the paradigmatic scenario of parameter estimation in quantum metrology [15] (see Fig. 3).

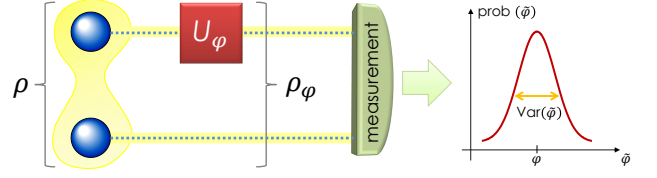


FIG. 3: (Color online) Quantum correlations-assisted parameter estimation. A probe state  $\rho$  of a bipartite system  $AB$  is prepared, and a local unitary transformation depending on an unobservable parameter  $\phi$  acts on subsystem A, transforming the global state into  $\rho_\phi$ . By means of a suitable measurement at the output one can construct an (unbiased) estimator  $\hat{\phi}$  for  $\phi$ . The quality of the estimation strategy is benchmarked by the variance of the estimator. For a given probe state  $\rho$ , the optimal measurement at the output returns an estimator for  $\phi$  with the minimum allowed variance given by the inverse of the quantum Fisher information  $\mathcal{F}(\rho_\phi)$ , according to the quantum Cramér-Rao bound [28]. In the prototypical case of optical phase estimation, the present scheme corresponds to a Mach-Zender interferometer. Restricting to pure inputs, research in quantum metrology [15] has shown that in this case entangled probes allow to beat the shot noise limit  $\mathcal{F} \propto n$  ( $n$  being the input mean photon number) and reach ideally the Heisenberg scaling  $\mathcal{F} \propto n^2$ . However, recent investigations have revealed how in presence of realistic imperfections the achieved precision quickly degrades to the shot noise level [30, 31]. For generally mixed probes, we show that the quantum Fisher information is bounded from below by the amount of quantum correlations in the probe state  $\rho$  as quantified by the LQU.

Given a bipartite state  $\rho$  used as a probe, subsystem A undergoes a unitary transformation (e.g. a phase shift) so that the global state changes to  $\rho_\phi = U_\phi \rho U_\phi^\dagger$ , where  $U_\phi = e^{-i\phi H_A}$ , with  $H_A$  a nontrivial local Hamiltonian on A. The goal is to estimate the unobservable parameter  $\phi$ . The protocol, which has wide-reaching applications, from gravimetry to sensing technologies [15, 27], can be optimised by picking the most informative measurement at the output and the best probe state  $\rho$ . It is known that the former optimisation can be solved in general by choosing, for any probe state  $\rho$ , the measurement strategy which saturates asymptotically the quantum Cramér-Rao bound,  $\text{Var}(\hat{\phi}) \geq 1/[\nu \mathcal{F}(\rho_\phi)]$  [28], where the quantum Fisher information  $\mathcal{F}(\rho_\phi)$  sets then the precision of the optimal estimation, and  $\nu$  denotes the number of times the experiment is repeated. Recall that the quantum Fisher information can be written as [27, 29]  $\mathcal{F}(\rho_\phi) = \text{Tr}\{\rho_\phi L_\phi^2\}$ , with  $L_\phi^2$  being the symmetric logarithmic derivative defined implicitly by  $2\partial_\phi \rho_\phi = L_\phi \rho_\phi + \rho_\phi L_\phi$ .

We focus therefore on the optimisation of the input state. In practical conditions, e.g. when the probing occurs within a thermal environment, it may not be possible to pick pure, maximally entangled probes. It is then of fundamental and practical importance to find out alternative resources enabling an enhancement in metrology within noisy settings [30, 31]. Here we assess whether and how quantum correlations in the (generally mixed) state  $\rho$  improve the sensitivity of the estimation. The key observation stems from the relation between the Wigner-Yanase and the Fisher metrics [29], which implies that the skew information of the Hamiltonian is majorized by

the quantum Fisher information [16, 32]. As  $H_A$  is not necessarily the most certain local observable, the LQU itself fixes a lower bound to the quantum Fisher information:

$$\mathcal{U}_A(\rho) \leq \mathcal{I}(\rho, H_A) = \mathcal{I}(\rho_\varphi, H_A) \leq \frac{1}{4}\mathcal{F}(\rho_\varphi).$$

Uncertainty is usually reputed to be a price to pay for exploiting the explicative and applicative power of quantum mechanics. Nevertheless, we established on rigorous footings that quantum correlations measured by LQU, though not necessary [33–35], are a universal sufficient resource to ensure, even in absence of entanglement, a fixed lower bound on the precision of optimal parameter estimation in realistic conditions. For fully classically correlated probes, the lower bound in Eq. (5) is zero and the quantum Fisher information can decrease arbitrarily, corresponding to an unboundedly large  $\text{Var}(\tilde{\varphi})$ . For probe states with any nonzero amount of quantum correlations, and for  $\nu \gg 1$  repetitions of the experiment, the optimal detection strategy which saturates the quantum Cramér-Rao bound produces an estimator with necessarily limited variance, scaling as

$$\text{Var}(\tilde{\varphi}) \leq \frac{1}{4\nu\mathcal{U}_A(\rho)}. \quad (5)$$

*Conclusions.*— In this Letter we reported on the manifold scenarios in which the quantum uncertainty on single observables takes centre stage. The exploration of this concept allowed us to define and investigate a general measure of bipartite quantum correlations [10], which is physically insightful, mathematically rigorous and analytically computable. Quantum correlations, in the form known as quantum discord [8, 9], manifest in the fact that any single local observable displays an intrinsic quantum uncertainty. This can tighten the Heisenberg bound on pairs of local measurements [1]. Discord in mixed probe states, measured by the local quantum uncertainty, is further proven to be an operational resource to enhance the guaranteed sensitivity of noisy quantum metrology [15]. This fundamental insight paves the way to the design of quantum correlations-enhanced sensors, of direct technological relevance for precision measurements in adverse conditions, where entanglement is absent. We believe worthwhile to substantiate in future work the promising uncovered connections between quantum mechanics, information geometry and complexity science [21, 28, 36] by addressing the role of quantum uncertainty, in particular induced by quantum correlations, in such contexts.

*Appendix: Proofs.*— We refer to [16–18] for a summary of the relevant properties of the skew information which constitute the main ingredients of the proofs. In a bipartite system  $AB$ , classically correlated states  $\rho_c$  with respect to measurements on  $A$ , also known as  $A$ -classically correlated states or classical-quantum states, are states with zero quantum discord on  $A$ . For these states there exists at least one set of projectors  $\{\Pi_i = \Pi_i^A \otimes \mathbb{I}^B\}$  such that  $\rho_c = \sum_i \Pi_i \rho_c \Pi_i$ . The  $A$ -classically correlated states take in general the form  $\rho_c = \sum_i p_i |i\rangle\langle i|_A \otimes \tau_{iB}$  with  $\{|i\rangle\}$  denoting an orthonormal basis for subsystem  $A$ .

To prove that  $A$ -classically correlated states have vanishing LQU  $\mathcal{U}_A$ , it is sufficient to define the observable  $K^\Pi = K_A^\Pi \otimes \mathbb{I}_B$  where  $K_A^\Pi$  is diagonal in the basis defined by  $\{\Pi_i^A\}$ , to obtain  $[\rho_c, K^\Pi] = 0$  which means  $\mathcal{U}_A(\rho_c) = \mathcal{I}(\rho_c, K^\Pi) = 0$ . On the other hand, a vanishing LQU ensures the existence of a local observable  $\tilde{K}^A$  such that  $\mathcal{I}(\rho, \tilde{K}^A) = 0$ . Hence  $\tilde{K}^A$  commutes with the density matrix, and we can diagonalise them simultaneously. Since the observable is assumed nondegenerate, its eigenvectors define a *unique* basis on  $A$  (up to phases), say  $\{|k_i\rangle\}$ . Then, an eigenvector basis for  $\tilde{K}_A$  will be simply  $\{|k_i\rangle_A \otimes |\phi_{ij}\rangle_B\}$ , and the state must necessarily be of the form  $\rho_{K_A} = \sum_i p_{ij} |k_i\rangle\langle k_i|_A \otimes |\phi_{ij}\rangle\langle\phi_{ij}|_B$ , which is a zero discord state. This proves that  $\mathcal{U}_A(\rho)$  vanishes if and only if  $\rho$  is an  $A$ -classically correlated state.

Let us now show that the LQU is invariant under local unitary transformations. We have

$$\begin{aligned} \mathcal{U}_A((U_A \otimes U_B)\rho(U_A \otimes U_B)^\dagger) \\ &= \min_{K^A} \mathcal{I}((U_A \otimes U_B)\rho(U_A \otimes U_B)^\dagger, K_A \otimes \mathbb{I}_B) \\ &= \min_{K^A} \mathcal{I}(\rho, (U_A \otimes U_B)^\dagger (K_A \otimes \mathbb{I}_B) (U_A \otimes U_B)) \\ &= \min_{K^A} \mathcal{I}(\rho, (U_A^\dagger K_A U_A) \otimes \mathbb{I}_B) = \mathcal{U}_A(\rho), \end{aligned}$$

as minimising over the local observables  $K^A$  is obviously equivalent to do it over the ones rotated by  $U_A$ .

We then note that the skew information  $\mathcal{I}(\rho, K^A)$  is contractive under completely positive and trace-preserving maps  $\Phi_B$  on  $B$ ,  $\mathcal{I}(\rho, K_A \otimes \mathbb{I}_B) \geq \mathcal{I}((\mathbb{I}_A \otimes \Phi_B)\rho, K_A \otimes \mathbb{I}_B)$ . Consequently, the LQU inherits this property. Denoting as  $\tilde{K}_A$  the most certain observable for  $\rho$ , we have  $\mathcal{U}_A(\rho) = \mathcal{I}(\rho, \tilde{K}_A \otimes \mathbb{I}_B) \geq \mathcal{I}((\mathbb{I}_A \otimes \Phi_B)\rho, \tilde{K}_A \otimes \mathbb{I}_B) \geq \mathcal{U}_A((\mathbb{I}_A \otimes \Phi_B)\rho)$ .

Finally, for pure states  $\rho = |\psi\rangle\langle\psi|$ , the LQU reduces to the variance of  $K^A$  minimised over all local observables  $K^A$ . In the Supplemental Material [26] we present a proof (which can be of independent interest) that such a quantity decreases monotonically under local operations and classical communication, so that the LQU, alias minimal local variance, reduces to an entanglement measure on pure states.

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## Supplemental Material

### Quantum uncertainty on a single observable

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#### Proof of LOCC monotonicity of LQU for pure states

**Lemma 1.** Consider a  $N$ -dimensional density matrix  $\rho$ , and the set  $\{K\}$  of all observables with fixed spectrum  $(\lambda_1, \dots, \lambda_N)$ . Then, the variance  $\text{Var}_\rho(K) \equiv \mathcal{V}(\rho, K) = \text{Tr}\{\rho K^2\} - \text{Tr}\{\rho K\}^2$  is minimised by an observable  $K_0$  commuting with  $\rho$ .

*Proof.* Working in the eigenbasis of the density matrix, one has the representation  $\rho = \text{diag}(p_1, \dots, p_N)$ . An observable in the considered set can be written as  $K = V \text{diag}(\lambda_1, \dots, \lambda_N) V^\dagger$ , where  $V$  is a unitary transformation. The variance of  $K$  on the state  $\rho$  reads ( $V_{ij} \equiv \langle i|V|j\rangle$ )

$$\mathcal{V}(\rho, K) = \sum_{i,j} p_i \lambda_j^2 |V_{ij}|^2 - \left( \sum_{i,j} p_i \lambda_j |V_{ij}|^2 \right)^2 \equiv \text{Tr}\{PB\} - [\text{Tr}\{QB\}]^2 \quad P_{ij} \equiv p_i \lambda_j^2, \quad Q_{ij} \equiv p_i \lambda_j, \quad B_{ij} \equiv |V_{ij}|^2. \quad (\text{A.1})$$

Note that  $B$  is a unistochastic matrix, and in fact, any unistochastic matrix is expressible as  $B_{ij} = |V_{ij}|^2$  for some unitary  $V$ . Hence, the problem of minimising the variance can be equivalently formulated as a minimisation of the right hand side of Eq. (A.1) over the set of unistochastic matrices. Since every unistochastic matrix is also bistochastic (but not vice-versa), one has, in general

$$\min_{\{K\}} \mathcal{V}(\rho, K) \geq \min_{B \in \mathcal{B}} [\text{Tr}\{PB\} - [\text{Tr}\{QB\}]^2], \quad (\text{A.2})$$

where  $\mathcal{B}$  is the set of all  $N \times N$  bistochastic matrices. One can now exploit the Birkhoff-von Neumann theorem, and express a generic bistochastic matrix as a convex sum of permutations of the form  $B = \sum_k q_k S_k$ , where the  $q_k$ 's are probabilities and  $\{S_k\}$  is the set of permutation matrices in dimension  $N$ , which has  $N!$  elements. Then,

$$\begin{aligned} \min_{B \in \mathcal{B}} [\text{Tr}\{PB\} - [\text{Tr}\{QB\}]^2] &= \min_{\{q_k\}} \left[ \sum_k q_k \text{Tr}\{PS_k\} - \left( \sum_k q_k \text{Tr}\{QS_k\} \right)^2 \right] \geq \min_{\{q_k\}} \sum_k q_k [\text{Tr}\{PS_k\} - [\text{Tr}\{QS_k\}]^2] \\ &\geq \sum_k q_k [\text{Tr}\{PS_{\min}\} - [\text{Tr}\{QS_{\min}\}]^2] = \text{Tr}\{PS_{\min}\} - [\text{Tr}\{QS_{\min}\}]^2, \end{aligned} \quad (\text{A.3})$$

where we have exploited the convexity of the square, and  $S_{\min}$  is a particular permutation that minimises the expression  $\text{Tr}\{PS_k\} - [\text{Tr}\{QS_k\}]^2$ . Such minimising permutation can always be found since  $\{S_k\}$  is a finite set. Noting that permutations are also unistochastic matrices, the above steps imply that the equality sign in Eq. (A.2) can be always achieved:

$$\min_{\{K\}} \mathcal{V}(\rho, K) = \text{Tr}\{PS_{\min}\} - [\text{Tr}\{QS_{\min}\}]^2 = \sum_i p_i \lambda_{\mathcal{P}(i)}^2 - \left( \sum_i p_i \lambda_{\mathcal{P}(i)} \right)^2, \quad (\text{A.4})$$

where  $\mathcal{P}$  indicates the permutation of the indices associated to the matrix  $S_{\min}$ . This implies that the variance is minimised by an observable of the form  $K_0 = \text{diag}(\lambda_{\mathcal{P}(1)}, \dots, \lambda_{\mathcal{P}(N)})$ , which clearly commutes with  $\rho$ .  $\square$

**Lemma 2.** Let  $d_{A,B} \equiv \dim(\mathcal{H}_{A,B})$ . Suppose that  $d_A \leq d_B$ . Under local operations on subsystem  $A$ , a globally pure state  $|\psi\rangle$  evolves within a subspace  $\mathcal{H}_A \otimes \mathcal{H}_B$ , where  $\mathcal{H}_B$  is a  $d_A$ -dimensional subspace of  $\mathcal{H}_B$ .

*Proof.* We can suppose that  $|\psi\rangle$  is in Schmidt form:

$$|\psi\rangle = \sum_j^{d_A} c_j |i_A\rangle |i_B\rangle, \quad (\text{A.5})$$



Clearly,  $|\psi\rangle \in \mathcal{H}_A \otimes \tilde{\mathcal{H}}_B$ , where  $\tilde{\mathcal{H}}_B$  is spanned by the  $d_A$  orthonormal vectors  $\{|i_B\rangle\}$ . A local operation on  $A$  is described via Kraus operators of the form  $M^A = M_A \otimes \mathbb{I}_B$ . Applying the operator on the state, one has:

$$M^A|\psi\rangle = \sum_j^{d_A} c_i(M_A|i_A\rangle)|i_B\rangle, \quad (\text{A.6})$$

which is still a vector with support in  $\mathcal{H}_A \otimes \tilde{\mathcal{H}}_B$ .  $\square$

**Corollary.** *When applying operations on  $A$  to a pure state, we can suppose  $d_A \geq d_B$ . A proof of monotonicity in this particular case will then be sufficient.*

**Lemma 3.** *Suppose  $d_A \geq d_B$ , and the (non-degenerate and equally spaced) spectrum of the  $A$ -observables is fixed as  $\sigma(K_A) = \{\lambda_1, \dots, \lambda_{d_A}\}$ . One has,*

$$\mathcal{U}_A(|\psi\rangle\langle\psi|) = \min_{K_B \in \mathcal{K}_B} \mathcal{I}(|\psi\rangle\langle\psi|, \mathbb{I}_A \otimes K_B), \quad (\text{A.7})$$

where  $\mathcal{K}_B$  is the set of  $B$ -observables whose  $d_B$  eigenvalues are non degenerate and are a subset of  $\sigma(K_A)$ :  $\sigma(K_B) = \{\mu_1, \dots, \mu_{d_B} | \mu_j \in \sigma(K_A), \mu_i \neq \mu_j (i \neq j)\}$ .

*Proof.* We start by noting that, in general,  $\mathcal{U}_A(|\psi\rangle\langle\psi|) \leq \min_{K_B \in \mathcal{K}_B} \mathcal{I}(|\psi\rangle\langle\psi|, \mathbb{I}_A \otimes K_B)$ . In fact, by rotating  $|\psi\rangle$  to the Schmidt form, we see that the variance of any observable  $K_B \in \mathcal{K}_B$  is achieved by an operator  $K_A$  on  $A$ . Given  $K^B$  such that  $K^B|\psi\rangle = \sum_{ij} c_i(K_B)_{ij}|i_A\rangle|j_B\rangle$ , it is sufficient to choose  $K^A$  such that  $K^A|\psi\rangle = \sum_{ij} c_i(K_A)_{ij}|j_A\rangle|i_B\rangle = \sum_{ij} c_i(K_A)_{ij}|i_B\rangle|j_A\rangle$ . The two operators clearly yield the same variance, since the labels  $A, B$  do not affect its calculation. Note that it is always possible to pick  $K^A$  in the above form since the operators on  $A$  restricted to a  $d_B$ -dimensional subspace can assume the same form as any operator in  $\mathcal{K}_B$ .

We now show that the inequality  $\mathcal{U}_A(|\psi\rangle\langle\psi|) \geq \min_{K_B \in \mathcal{K}_B} \mathcal{I}(|\psi\rangle\langle\psi|, \mathbb{I}_A \otimes K_B)$  is also verified, hence equality must hold. The most certain observable on  $A$  has to commute with the reduced state  $\rho_A$  (Lemma 1). Hence, if the latter has eigenvalues  $p_j$ ,  $j \leq d_B$ , there is an appropriate permutation  $\mathcal{P}$  such that:

$$\mathcal{U}_A(\psi) = \sum_{j=1}^{d_B} p_j (\lambda_{\mathcal{P}(j)})^2 - \left( \sum_{j=1}^{d_B} p_j \lambda_{\mathcal{P}(j)} \right)^2 = \mathcal{I}(|\psi\rangle\langle\psi|, \mathbb{I}_A \otimes \tilde{K}_B). \quad (\text{A.8})$$

The latter equality is obtained by choosing  $\tilde{K}_B$  diagonal in the same basis as  $\rho_B$ , with eigenvalues  $\mu_j = \lambda_{\mathcal{P}(j)}$ , and by noting that  $\rho_A$  and  $\rho_B$  have the same eigenvalues.  $\square$

**Theorem.** *The LQU is an entanglement monotone for pure states.*

*Proof.* By Lemma 2, we can suppose  $d_A \geq d_B$ . We already have invariance under local unitaries and contractivity under local operations on  $B$  (see Appendix in the main text). To complete the proof we need to prove that, on average, the LQU of  $|\psi\rangle$  cannot be increased under operations on  $A$ . Let  $\{M_i^A\}$  be the Kraus operators on Alice:  $\sum_i M_i^{A\dagger} M_i^A = \mathbb{I}$ . The output ensemble is given by  $\{p_i, |\phi_i\rangle\}$ , where

$$\sqrt{p_i}|\phi_i\rangle = M_i^A|\psi\rangle. \quad (\text{A.9})$$

We want to demonstrate that  $\sum_i p_i \mathcal{U}_A(|\phi_i\rangle\langle\phi_i|) \leq \mathcal{U}_A(|\psi\rangle\langle\psi|)$ . Suppose that  $K_0 \in \mathcal{K}_B$  is such that  $\mathcal{U}_A(|\psi\rangle\langle\psi|) = \mathcal{I}(|\psi\rangle\langle\psi|, \mathbb{I}_A \otimes K_0)$ , as given by Lemma 3.

$$\begin{aligned} \sum_i p_i \mathcal{U}_A(|\phi_i\rangle\langle\phi_i|) &= \sum_i p_i \min_{K_i \in \mathcal{K}_B} \mathcal{I}(|\phi_i\rangle\langle\phi_i|, \mathbb{I}_A \otimes K_i) \leq \sum_i p_i \mathcal{I}(|\phi_i\rangle\langle\phi_i|, \mathbb{I}_A \otimes K_0) \\ &= \sum_i p_i \mathcal{V}(|\phi_i\rangle\langle\phi_i|, \mathbb{I}_A \otimes K_0) \leq \mathcal{V}(\sum_i p_i |\phi_i\rangle\langle\phi_i|, \mathbb{I}_A \otimes K_0) \\ &= \sum_i p_i \langle\phi_i| \mathbb{I}_A \otimes K_0^2 |\phi_i\rangle - (\sum_i p_i \langle\phi_i| \mathbb{I}_A \otimes K_0 |\phi_i\rangle)^2 \\ &= \sum_i \langle\psi| M_i^A (\mathbb{I}_A \otimes K_0^2) M_i^{A\dagger} |\psi\rangle - \left( \sum_i \langle\psi| M_i^A (\mathbb{I}_A \otimes K_0) M_i^{A\dagger} |\psi\rangle \right)^2 \\ &= \langle\psi| \sum_i M_i^{A\dagger} M_i^A \otimes K_0^2 |\psi\rangle - \left( \langle\psi| \sum_i M_i^{A\dagger} M_i^A \otimes K_0 |\psi\rangle \right)^2 \\ &= \langle\psi| \mathbb{I}_A \otimes K_0^2 |\psi\rangle - (\langle\psi| \mathbb{I}_A \otimes K_0 |\psi\rangle)^2 = \mathcal{I}(|\psi\rangle\langle\psi|, \mathbb{I}_A \otimes K_0) = \mathcal{U}_A(|\psi\rangle\langle\psi|). \end{aligned} \quad (\text{A.10})$$

In the first line, we used Lemma 3. In the second line, we have used that the variance is concave as a function of the state.  $\square$

### Example: LQU in the DQC1 model

An interesting case study concerns the final state of the DQC1 (Discrete Quantum Computation with One bit) model, a protocol designed for estimating the trace of a unitary matrix, say  $U$ , applied on a  $n$ -qubit register [25]. Discord-like correlations, but vanishing entanglement, are created between an ancillary qubit and the register in the output state [11]. The ancilla  $A$ , in a state with arbitrary polarisation  $\mu$ , say  $\rho_A^{in} = \frac{1}{2}(\mathbb{I}_2 + \mu\sigma_3)$ , and the register  $B$ , in a  $n$ -qubit maximally mixed state, i.e.,  $\rho_B^{in} = \frac{1}{2^n}\mathbb{I}_n$ , are initially uncorrelated:  $\rho^{in} = \rho_A^{in} \otimes \rho_B^{in}$ . The protocol returns the final state

$$\rho^{out} = \frac{1}{2^{n+1}} \left( \begin{array}{c|c} \mathbb{I}_n & \mu U^\dagger \\ \hline \mu U & \mathbb{I}_n \end{array} \right). \quad (\text{A.11})$$

Measuring the ancilla polarisation in the output state yields an estimation of the trace of the unitary matrix:  $\langle \sigma_1 \rangle_{\rho_A^{out}} = \text{Re}[\text{Tr}[U]]$ ,  $\langle \sigma_2 \rangle_{\rho_A^{out}} = \text{Im}[\text{Tr}[U]]$ . For ‘typical’ unitaries in high dimensions (which have approximately zero trace [11]), the entanglement between the ancilla and the  $n$ -qubit register is always negligible. On the contrary we find the following.

**Proposition.** *The local quantum uncertainty calculated via Eq. (3) yields:*

$$\mathcal{U}_A(\rho^{out}) = \frac{1}{2} \left( 1 - \sqrt{1 - \mu^2} \right). \quad (\text{A.12})$$

*Proof.* We choose the basis  $\{|k\rangle\}$  on  $B$  which diagonalises  $U$ :  $U|k\rangle = e^{-i\varphi_k}|k\rangle$ . We may then rewrite Eq. (A.11) as  $\rho^{out} = 2^{-n} \sum_k \rho_k \otimes |k\rangle\langle k|$ , where  $\rho_k = 1/2(\mathbb{I}_2 + \vec{\mu}_k \cdot \vec{\sigma})$  and  $\vec{\mu}_k = \mu(\cos \varphi_k, \sin \varphi_k, 0)$ . The square root of the density matrix can then be expressed as  $\sqrt{\rho^{out}} = 2^{-n/2} \sum_k r_k \otimes |k\rangle\langle k|$ , where  $r_k = 2^{-1/2}(v_0\mathbb{I}_2 + \vec{v}_k \cdot \vec{\sigma})$ , where  $\vec{v}_k = v(\cos \varphi_k, \sin \varphi_k, 0)$  and the pair  $v_0, v$  verify  $v_0^2 + v^2 = 1$  and  $2v_0v = \mu$ . Both  $v_0$  and  $v \equiv |\vec{v}_k|$  do not depend on  $k$ , while  $\vec{v}_k$  does. The elements of the matrix  $W_{AB}$  are then given by

$$\begin{aligned} (W_{AB})_{ij} &= \frac{1}{2^n} \sum_k \text{Tr}\{r_k \sigma_i r_k \sigma_j\} \\ &= v_0^2 \delta_{ij} + 2^{-(n+1)} \sum_{k,l,m} (\vec{v}_k)_l (\vec{v}_k)_m \text{Tr}\{\sigma_i \sigma_l \sigma_j \sigma_m\}. \end{aligned} \quad (\text{A.13})$$

Now, we see that  $\text{Tr}\{\sigma_i \sigma_l \sigma_j \sigma_m\} = 2(\delta_{il}\delta_{jm} - \delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl})$ . Hence,

$$(W_{AB})_{ij} = (v_0^2 - v^2)\delta_{ij} + \frac{2}{2^n} \sum_k (\vec{v}_k)_i (\vec{v}_k)_j. \quad (\text{A.14})$$

Substituting the explicit expressions for the components of  $\vec{v}_k$ , Eq. (A.14) requires evaluation of the sums  $2^{-n} \sum_k \cos^2 \varphi_k$ ,  $2^{-n} \sum_k \sin^2 \varphi_k$ , and  $2^{-n} \sum_k \sin \varphi_k \cos \varphi_k$ . We observe that for large  $n$  and ‘typical’ unitaries, where the phases  $\varphi_k$  are uniformly distributed [11], we can approximate those sums with integral averages of the trigonometric functions over the interval  $\varphi \in [0, 2\pi]$ :  $\langle \cos^2 \rangle \simeq \langle \sin^2 \rangle \simeq 1/2$ ,  $\langle \sin \cos \rangle \simeq 0$ . Then,

$$W_{AB} \simeq \text{diag}\{v_0^2, v_0^2, v_0^2 - v^2\} \Rightarrow \lambda_{\max}(W_{AB}) = v_0^2. \quad (\text{A.15})$$

Finally, the conditions given above on  $v_0, v$  can be used to express  $v_0$  in terms of the qubit initial polarisation, as  $v_0^2 = 1/2(1 + \sqrt{1 - \mu^2})$ . Substituting this in Eq. (3) yields the anticipated result of Eq. (A.12). As expected, the expression increases monotonically with the ancilla polarisation and is independent of the number of qubits in the register. This is in agreement with what predicted by using the quantum discord [8, 11].  $\square$